First Virtual ECMI Modeling Week 2020
Peter the Great St.Petersburg Polytechnic University

# Capillary moisture uptake in trees

Świtała Mateusz (Wrocław University of Science and Technology)

Ba Fatima Antarou (Technical University of Berlin)
Herde Alexander (Technical University of Berlin)
Hou Kaiwen (Tsinghua University)
Rapajić Isidora (University of Novi Sad)

July 2020

# Contents

# 1 Introduction

Water plays a crucial role in the growth and photosynthesis in plants, as well as in the transport of organic and inorganic molecules through the wood tissues. By roots water in trees is absorbed from the soil. When absorbed, water crosses several cell layers before entering the water transport tissue (xylem - which has a diameter from 10 to 100 $\mu$m in trees). Once in the xylem, water moves easily over long distances. There are two kinds of transport tubes in the xylem: tracheids and vessels. Tracheids are smaller than vessels in both diameter and length. (Tyree and Zimmermann, 2002) When the water reaches the end of a tube, to pass to an adjacent one, it has to cross through the pits in the cell walls (pit membrane).

Cohesion-Tension theory (Dixon and Joly, 1894) is the most widely accepted explanation for water transport in trees. It states that the evaporation of water from leaves (transpiration) causes negative pressure so that the water is pulled into the leaf from the xylem. The assumption is that water is cohesive - which means that the water flow does not break due to hydrogen bonding. The pulling of water (suction tension) will extend down through the rest of the xylem pipeline of the tree and into the xylem of the roots due to cohesive forces.

In the following report, the water uptake in wood is modeled by capillary action, which is the tendency of water to rise in a narrow tube without the assistance of external forces. In terms of physics, concave meniscus forms when the lower end of a narrow pipe is placed in a container with liquid. Adhesive forces occur between the liquid and an inner wall of a pipe, which results in uptake of a liquid until its mass and intermolecular forces are sufficient to overcome gravitational forces. Capillary rise (Jurin's law) is the most basic analysis of capillary action. It suggests that the equilibrium height (Jurin's height) of a liquid in a capillary tube is inversely proportional to the tube's diameter. It is necessary to take into consideration that xylem tissue is composed of elongated cells that are connected, forming a transport pipeline from roots to leaves. In literature, one can find evidence that the water uptake in plants and even in cells of some small animals is based on capillary action (Bentley and Blumer, 1962).

Our goal is to model the capillary moisture uptake in wood and trees as well as the moisture balance in forest. In Chapter 2, the topics are the models of capillary moisture uptake in wood, trees, and moisture balance in a forest. Chapter 3 will feature conclusions and discussion. Work dynamics of the group and instructor assessment are presented in Chapters 4 and 5, respectively.

The gathered results of capillary moisture uptake in wood and trees could provide some new insights into the internal structure of wood. Furthermore, the model of moisture balance in forests could be meaningful for ecological analysis.

# 2 Report content

## 2.1 Capillary moisture uptake in wood

Building on the investigated model that assumes that a single capillary is vertically immersed in a tank filled with water, the general model of capillary water absorption in wood can be derived on the assumption that the internal structure of wood consists of a bundle of capillaries with different diameters. The governing equation that models the water rise in a single narrow tube with the circular cross-section can be derived using Newton's second law

$$\begin{cases} (\rho(h(t)+h_0)h'(t))' + (1+C)\dfrac{8\mu}{r^2}(h(t)+h_0)h'(t) + \rho g h(t) = \dfrac{2\gamma\cos(\theta)}{r}, \\ h(0)=0, h'(0)=0, \end{cases} \tag{U}$$

where $h = h(t)$ is the liquid column height at time $t$, $h_0$ is the immersion length of the tube below the free surface of liquid outside the tube, $C$ represents the structural resistance caused mainly by the pits which join two successive vertical xylem tissues, $\mu$ is the dynamic viscosity of the water in $[\mathrm{kg}/(\mathrm{m}\cdot\mathrm{s})]$, $\gamma$ is the surface tension in $[\mathrm{N/m}]$, $\theta$ is the contact angle of the meniscus with the wood tissue wall in degree, $g$ is the gravitational acceleration in $[\mathrm{m/s^2}]$, $\rho$ the water density in $[\mathrm{kg/m^3}]$ and finally $r$ is the radius of the xylem pipes in [m], which is in general between $10^{-5}$ and $10^{-4}$m. The immersion length $h_0$ is a length of water column inside the tube below the initial height $h(0)$ and is assumed to be of the order of tube radius. Including the immersion length in the following analysis implies that the initial state of the capillary rise in the investigated problem is slightly above the tube entrance. Note that for $h_0 = 0$ the equation (U) becomes singular what may make the theoretical and numerical analysis of the problem more difficult. The first term on the left-hand side of the equation (U) represents the inertia of the liquid inside the tube, the second term can be derived from the Hagen-Poiseuille equation and denotes the energy loss by water due to the viscosity. The last term on the left is hydrostatic pressure. Finally, the right-hand side of (U) describes the capillary pressure, i.e. the pressure which is responsible for the spontaneous liquid rise in a narrow vertical tube. The exact form of the capillary pressure can be obtained using the Young-Laplace equation. Note that is important to include the constant $C$ in the model since in contrast to ordinary cylindrical pipes with smooth walls, the internal structure of wood contains the pit membranes that connect the consecutive xylem tissues and hence may exert an additional resistance against the water flow. Furthermore, the xylem tissue walls are not completely smooth what also may cause additional energy loss during the flow. Hence, the parameter $C$ in the model (U) is responsible for the wood structural resistance.

### 2.1.1 Structural resistance

As was stated above, the second term of the governing equation can be directly derived from the Hagen-Poiseuille equation, which describes a pressure drop or an "energy loss" due to the friction in a sufficiently long cylindrical pipe. The Hagen-Poiseuille equation is derived under the assumption that the fluid is incompressible and Newtonian, the flow is laminar (parabolic velocity profile) and the cross-section of a pipe is constant. The Hagen-Poisseuille equation is commonly expressed in the following form

$$\Delta p = \frac{8\mu}{\pi r^4} LQ, \tag{1}$$

where $L$ is a pipe length and $Q$ is a volumetric flow rate.

The equation generated by a velocity profile associated with a laminar flow that assumes that the highest velocity is in the middle of a pipe is

$$v(t,x) = 2h'(t)\left(1 - \frac{x^2}{r^2}\right), \; x \in [0,r]. \tag{2}$$

And thus the volumetric flow rate $Q$ can be calculated using $A = \pi x^2$ and $d\mathbf{A} = 2\pi x dx$

$$Q = \iint_A \mathbf{v} \cdot d\mathbf{A} = 2 \cdot 2\pi \int_0^r h'(t)x\left(1 - \frac{x^2}{r^2}\right)dx = \pi r^2 \cdot h'(t). \tag{3}$$

The final form of $\Delta p$ is a result of substituting $L = h(t)$ and $Q = \pi r^2 \cdot h'(t)$ into the equation

$$\Delta p = \frac{8\mu}{\pi r^4} LQ = \frac{8\mu}{\pi r^4} h \cdot Q = \frac{8\mu}{\pi r^4} hh' \cdot \pi r^2 = \frac{8\mu}{r^2} hh'. \tag{4}$$

The given model is not sufficiently accurate since it works under assumptions that do not reflect reality in xylem conduits.

After taking into consideration the structure of the capillary tubes in wood, in particular the fact that the walls of wood capillaries are not completely smooth and that two successive conduits are connected through a pit membrane, it is apparent that the employed model has to be ameliorated. The authors include a new collective term to outline the differences between the core assumptions of the Hagen-Poiseuille equation for cylindrical pipes and capillary rise in wood and put all of those into one common component, called structural resistance $C$.

$$\Delta p = (1 + C)\frac{8\mu}{r^2} hh'. \tag{5}$$

A more comprehensive analysis of the structural resistance factor is required to find the most significant features of flow rate-determining xylem conduits and therefore provide an accurate formula for $C$ that may be useful in applications. The authors of this article conjecture that the length of wood tissues may be one of these factors, i.e. the shorter the conduit length is, the more frequently pits between two successive capillaries appear, and thus water flowing through multiple pit membranes is exposed to the additional resistance during the flow.

### 2.1.2 Analysis of a structural resistance factor

Since the cross-section of the narrow tube in the investigated model is assumed to be constant and its diameter is much less than unity, only one-dimensional vertical flow is considered. The initial height $h(0)$ and velocity $h'(0)$ are assumed to be equal to zero. To facilitate analysis let us introduce a new variable $H(t) := h(t) + h_0$. After substituting $H(t)$ into (U) the governing equation is

$$\begin{cases} (\rho H(t)H'(t))' + (1+C)\frac{8\mu}{r^2}H(t)H'(t) + \rho g\left(H(t) - h_0\right) = \frac{2\gamma\cos(\theta)}{r}, \\ H(0) = h_0, H'(0) = 0, \end{cases} \tag{W}$$

Because the structural resistance factor has to be determined experimentally, so now the task is to find the exact mathematical formula for estimating $C$ for a particular wood species, so that such an analysis can be used in various applications. Since the process of water rise in xylem conduits is extremely complex, the expression for structural resistance $C$ cannot be obtained in an explicit form. Nevertheless, as will be shown, the approximate value $\hat{C}$ of the structural resistance factor can be calculated using only some experimental data and the physical quantities that characterize the flow. In the subsequent sections, it is shown that $\hat{C}$ can be expressed in terms of known variables $\rho, \mu, r$, the length of the considered piece of wood $\hat{L}$, and the time it takes the water to reach its top $\hat{t}$ . In general, the $\hat{L}$ may represent any level of water in the sample reported in the experiment, and then $\hat{t}$ will be the first time after which the height $\hat{L}$ has been reached by water.

### 2.1.3 Neglecting the gravity term - inertia and viscosity regime

To simplify the problem, it is assumed that the gravitational component can be neglected at the beginning of the flow when inertia and viscosity forces are dominant. However, it is necessary to include the gravitational component at the very end of the pipe (near the Jurin's height) to balance the capillary pressure and therefore stop the rise of water. After careful examination of the model, it turns out that the influence of the inertia term near the equilibrium height is negligible, and thus it is justifiable to consider only the viscosity and gravitational forces as resistance forces acting on the water column when the height of the liquid is close to equilibrium height.

To check whether ignoring the gravity term is reasonable for the given parameters and in an appropriate time range, the model will be transformed to dimensionless form. First, divide both sides of *(W)* by capillary pressure

$$\frac{r\left(\rho H(t)H'(t)\right)'}{2\gamma\cos(\theta)} + (1+C)\frac{4\mu}{r\gamma\cos(\theta)}H(t)H'(t) + \frac{\rho g r\left(H(t) - h_0\right)}{2\gamma\cos(\theta)} = 1. \tag{6}$$

Next, let us introduce non-dimensional height and time

$$\mathcal{H} = \frac{H}{L}, \quad s = \frac{t}{T}, \tag{7}$$

and consequently, $\mathcal{H}_0 = h_0/L$, where $L$ and $T$ are reference height and reference time respectively. After substitution, the equation changes to

$$\frac{r\rho}{2\gamma\cos(\theta)}\frac{L^2}{T^2}\frac{d}{ds}\left(\mathcal{H}\frac{d\mathcal{H}}{ds}\right) + \frac{4\mu(1+\hat{C})}{r\gamma\cos(\theta)}\frac{L^2}{T}\left(\mathcal{H}\frac{d\mathcal{H}}{ds}\right) + \frac{\rho g r}{2\gamma\cos(\theta)}L(\mathcal{H}-\mathcal{H}_0) = 1. \quad (8)$$

Next, assuming that components in front of viscosity and inertia term are of the order of unity, and for simplicity, assuming that they are equal one, the resulting equation is reduced to

$$\frac{r\rho}{2\gamma\cos(\theta)}\frac{L^2}{T^2} = 1, \qquad \frac{4\mu(1+\hat{C})}{r\gamma\cos(\theta)}\frac{L^2}{T} = 1.$$

Thus, it holds

$$T = \frac{r^2\rho}{8\mu(1+\hat{C})} < \frac{r^2\rho}{8\mu}, \qquad L = \frac{r}{4\mu(1+\hat{C})}\sqrt{\frac{r\rho\gamma\cos(\theta)}{2}} < \frac{r}{4\mu}\sqrt{\frac{r\rho\gamma\cos(\theta)}{2}}. \quad (9)$$

The equation (8) can be then simplified as follows

$$\frac{d}{ds}\left(\mathcal{H}\frac{d\mathcal{H}}{ds}\right) + \left(\mathcal{H}\frac{d\mathcal{H}}{ds}\right) + \varepsilon(\mathcal{H}-\mathcal{H}_0) = 1, \quad (10)$$

where

$$\varepsilon = \frac{\rho g r}{2\gamma\cos(\theta)}\cdot\frac{r}{4\mu(1+\hat{C})}\sqrt{\frac{r\rho\gamma\cos(\theta)}{2}} < \frac{\rho g r}{2\gamma\cos(\theta)}\cdot\frac{r}{4\mu}\sqrt{\frac{r\rho\gamma\cos(\theta)}{2}}. \quad (11)$$

Note that equation (9) and (11) hold since the structural resistance is positive which in turn implies $(1+\hat{C}) > 1$. By taking $\rho = 997\,\mathrm{kg/m^3}$, $g = 9.81\,\mathrm{m/s^2}$, $r = 10^{-5}\,\mathrm{m}$, $\mu = 10^{-3}\,\mathrm{Pa\cdot s}$, $\theta = 0$, $\gamma = 72.8\cdot 10^{-3}\,\mathrm{N/m}$ one gets that

$$T < 1.25\cdot 10^{-5}\,\mathrm{s}, \; L < 4.76\cdot 10^{-5}\,\mathrm{m}, \; \varepsilon < 3.2\cdot 10^{-5}. \quad (12)$$

Since the coefficient in front of the gravity term ($\varepsilon$) is much smaller than the viscosity and inertia term (both of the order of unity), neglecting the gravity term it will not significantly change the exact solution.

#### 2.1.4 Neglecting the inertia term - viscosity and gravity regime

The inertia term from the governing equation in its dimensionless form (equation (8)) seems to be negligible at the very end of a flow. Therefore only the viscosity and gravity terms are of relevance. Applying a similar procedure as in the last section 2.1.3, that is assuming that components before the gravity and viscosity term are of the order of unity

$$\frac{4\mu(1+\hat{C})}{r\gamma\cos(\theta)}\frac{L^2}{T} = 1, \qquad \frac{\rho g r}{2\gamma\cos(\theta)}L = 1,$$

one then gets by solving the system of equations that

$$T = \frac{16\mu\gamma\cos(\theta)(1+\hat{C})}{r^3\rho^2 g^2} > \frac{16\mu\gamma\cos(\theta)}{r^3\rho^2 g^2}, \quad L = \frac{2\gamma\cos(\theta)}{\rho g r}.$$

Hence equation (8) can be rewritten with only one non-dimensional parameter,

$$\varepsilon\frac{d}{ds}\left(\mathcal{H}\frac{d\mathcal{H}}{ds}\right) + \mathcal{H}\frac{d\mathcal{H}}{ds} + \mathcal{H} - \mathcal{H}_0 = 1, \tag{13}$$

where

$$\varepsilon = \frac{r\rho}{2\gamma\cos(\theta)}\frac{L^2}{T^2} = \frac{r^5\rho^3 g^2}{128\mu^2\gamma\cos(\theta)(1+\hat{C})^2} < \frac{r^5\rho^3 g^2}{128\mu^2\gamma\cos(\theta)}.$$

Similarly to the first case, by plugging the exact values of physical quantities into $T$ and $L$, one gets

$$T > 12\cdot 10^3\,\mathrm{s},\ \ L \approx 1.49\,\mathrm{m},\ \ \varepsilon < 1.02\cdot 10^{-9}. \tag{14}$$

Since the coefficient in front of the inertial term $\varepsilon$ is much smaller than both viscosity and gravity terms (order of unity), it may be neglected.

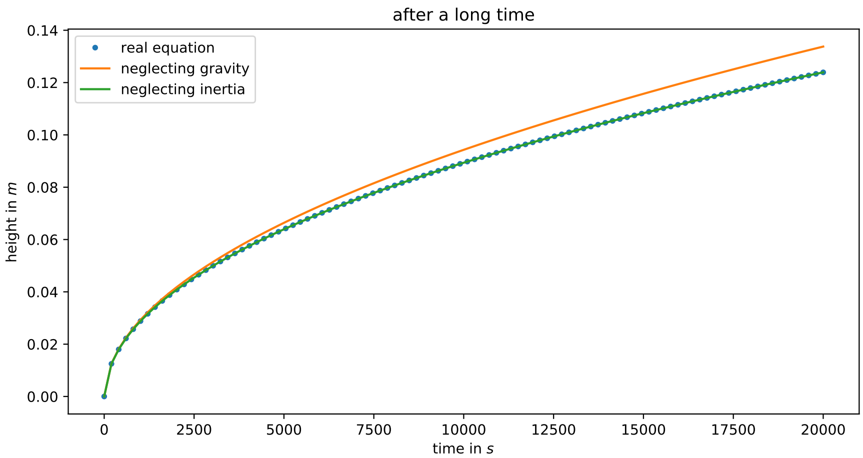


Figure 1: Numerical comparison between solution to (W) and a simplified version for vessels in a long time ($C = 1000$)

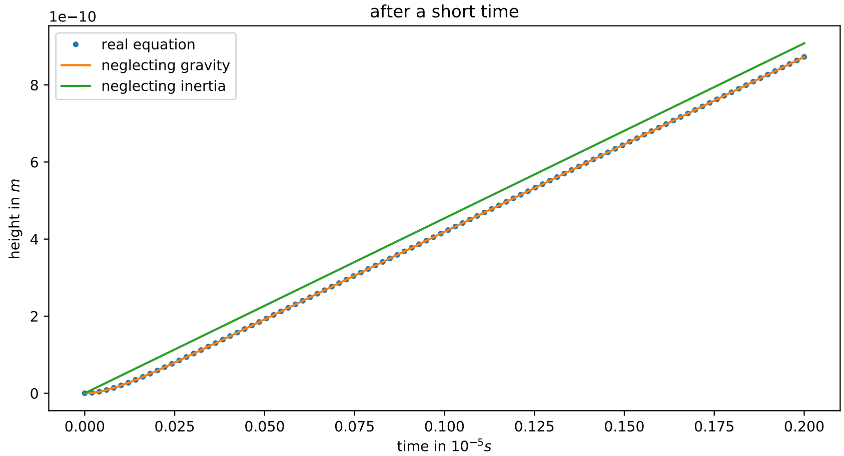


Figure 2: Numerical comparison between solution to (W) and simplified version for vessels in a short time ($C = 1000$)

Following the analysis that has been carried out in both sections 2.1.3 and 2.1.4, note that for sufficiently small values of the time variable, one can neglect the gravity term to obtain an approximate solution of the initial value problem (U). However, this hypothesis is no longer valid when the value of the time-variable is large enough, or in other words, when the liquid height is near the Jurin's height. Indeed from there on, one must instead ignore the inertia term to obtain a valid approximation of the exact solution. It has been shown numerically that the resulting error is relatively small in both cases, as illustrated in Figs. 1 and 2. The approximate solution resulting from disregarding the gravitational term has the same asymptotic behavior as the solution of the original differential equation when the time is of the order of $10^{-5}$ s and smaller, and the solution resulting from dismissing the inertia term in the governing equation converges towards the exact solution when the time is of the order of $10^3$ s and greater.

### 2.1.5 Estimating structural resistance parameter (gravity neglected)

In the following, an estimation of the structural resistance is calculated by first looking for a solution of the governing differential equation (W). For this, one further uses notation $\hat{C}$ in the simplified equations, which represents an estimation of the exact value of $C$, that is

$$(\rho H(t)H'(t))' + (1+\hat{C})\frac{8\mu}{r^2}H(t)H'(t) = \frac{2\gamma\cos(\theta)}{r}.$$

In addition to the initial conditions, $h(0) = 0,\ h'(0) = 0$, one assume that the time $\hat{t}$ required to reach some height $\hat{L}$ is known, for example, by determining it during an experiment, one is then able to estimate $\hat{C}$. To simplify the computation, define a new variable $z$ as:

$$z(t) := \frac{\big(H(t)\big)^2}{2} \text{ and } z'(t) = \left(\frac{\big(H(t)\big)^2}{2}\right)' = H(t)H'(t), \tag{15}$$

such that

$$\begin{cases} \rho z''(t) + (1+\hat{C}) \cdot \frac{8\mu}{r^2} z'(t) = \frac{2\gamma\cos(\theta)}{r}, \\ z(0) = \dfrac{h_0^2}{2}, z'(0) = 0, \\ z(\hat{t}) = \dfrac{\hat{L}^2}{2}. \end{cases} \tag{$W_s$}$$

By assuming that Jurin's height or the equilibrium height $h_e$ is an upper boundary, it is known that the height of the water within the capillaries $h_w$ is smaller than or equal $h_e$: $h_w \leq h_e$. Since $\hat{C}$ in $(W_s)$ is expected to depends mainly on the experimentally determined quantities $\hat{L}$ and $\hat{t}$ , then, without loss of generality, the $\hat{C}$ can be treated as a constant and then solve this linear second-order non-homogeneous ODE with constant coefficients.

The solution of the problem $(W_s)$ is given by the sum of the general solution to the homogeneous problem defined as

$$\rho z_h''(t) + (1+\hat{C}) \cdot \frac{8\mu}{r^2} z_h'(t) = 0, \tag{16}$$

and a particular solution denoted by $z_p(t)$ to the non-homogeneous problem. The solution $z_h(t)$ of the homogeneous problem (16) is obtained by determining the roots $w_1$ and $w_2$ of its corresponding second-order characteristic polynomial. One can easily verify that the solution for the homogeneous problem is

$$z_h(t) = k_1 e^{w_1 t} + k_2 e^{w_2 t} = k_1 + k_2 e^{-\frac{8\mu}{\rho r^2}(1+C)\cdot t}, \tag{17}$$

where $k_1$ and $k_2$ are the constants that will be computed later by using the initial conditions.

The particular solution $z_p(t)$ of the non-homogeneous problem $(W_s)$ is given by

$$z_p(t) = \frac{2\gamma r \cdot \cos(\theta)}{8\mu(1+\hat{C})} t. \tag{18}$$

It follows that the general solution of the non-homogeneous problem $(W_s)$ is

$$\begin{aligned} z(t) &= z_h(t) + z_p(t) \\ &= k_1 + k_2 e^{w_2 t} + \frac{2\gamma r \cdot \cos(\theta)}{8\mu(1+\hat{C})} t \\ &= k_1 + k_2 e^{-\frac{8\mu}{\rho r^2}(1+\hat{C})t} + \frac{2\gamma r \cdot \cos(\theta)}{8\mu(1+\hat{C})} t. \end{aligned}$$

To compute the coefficients $k_1$ and $k_2$ the appropriate initial conditions are used (see $(W_s)$)

$$\begin{cases} \frac{h_0^2}{2} = z(0) = k_1 + k_2, \\ 0 = z'(0) = -\frac{8\mu}{\rho r^2}(1+\hat{C})k_2 + \frac{2\gamma r \cdot}{8\mu(1+\hat{C})}, \end{cases} \tag{19}$$

since the first derivative of the solution is equal to

$$z'(t) = -\frac{8\mu}{\rho r^2}(1+\hat{C})k_2 e^{-\frac{8\mu}{\rho r^2}(1+\hat{C})t} + \frac{2\gamma r \cdot \cos(\theta)}{8\mu(1+\hat{C})}. \tag{20}$$

By solving the system of linear equations (19) one gets

$$k_1 = \frac{h_0^2}{2} - \frac{2\gamma\rho r^3 \cdot \cos(\theta)}{(8\mu)^2(1+\hat{C})^2} \text{ and } k_2 = \frac{2\gamma\rho r^3 \cdot \cos(\theta)}{(8\mu)^2(1+\hat{C})^2}. \tag{21}$$

Hence the general solution of the problem $(W_s)$ is equal to

$$z(t) = \frac{h_0^2}{2} - \frac{2\gamma\rho r^3 \cdot \cos(\theta)}{(8\mu)^2(1+\hat{C})^2} + \frac{2\gamma\rho r^3 \cdot \cos(\theta)}{(8\mu)^2(1+\hat{C})^2} e^{-\frac{8\mu}{\rho r^2}(1+\hat{C})t} + \frac{2\gamma r \cdot \cos(\theta)}{8\mu(1+\hat{C})} t. \tag{22}$$

The final solution $h(t)$ is derived by plugging $z(t)$ into (15)

$$h(t) = \sqrt{2z(t)} - h_0, \text{ and for all } t \geq 0, \tag{23}$$

since $z(t) > 0$ for all $t \in [0, \hat{t}]$ (for radius $r$ of the order of $10^{-5}$ m). Since the exponential term $\exp(-8\mu/\rho r^2(1+\hat{C})t)$ converges rapidly to zero for growing $t$ so neglecting it in the following analysis will not affect the solution significantly. Indeed, after substituting the exact value of water density, $\rho = 997\,\mathrm{kg/m^3}$, narrow radius, $r = 10^{-5}\,\mathrm{m}$, and dynamic viscosity, $\mu = 10^{-3}\,\mathrm{kg/(m \cdot s)}$, into an exponential term one gets

$$\frac{8\mu}{\rho r^2}(1+\hat{C}) \geq 80240.7 \text{ s}^{-1}] \gg 1 \text{ [s}^{-1}, \tag{24}$$

since $\hat{C}$ is a positive quantity and hence $1+\hat{C} \geq 1$. Note that the exponent becomes significantly large even for small $t$ values what immediately indicates that the whole term becomes small in comparison to other components in (22). Now, using the auxiliary condition from $(W_s)$, i.e. $z(\hat{t}) = \hat{L}^2/2$, and after some algebraic manipulation, the exact formula for $\hat{C}$ has been found

$$\hat{C}_{\pm} = \frac{-\gamma r\hat{t}\mu\cos(\theta) \pm r\mu\sqrt{\cos(\theta)(\gamma^2\hat{t}^2\cos(\theta) - \gamma\rho r(h_0^2+\hat{L}^2))}}{4\mu^2(h_0^2+\hat{L}^2)}. \tag{25}$$

By setting the end value condition $\hat{t} = 1\,\text{day} = 86400\,\text{s}$ and the high $\hat{L} = 0.1\,\text{m}$ and also taking $\rho = 997\,\mathrm{kg/m^3}$, $g = 9.81\,\mathrm{m/s^2}$, $r = 10^{-5}\,\mathrm{m}$, $\mu = 10^{-3}\,\mathrm{Pa \cdot s}$, $\theta = 0$, $\gamma = 72.8 \cdot 10^{-}3\,\mathrm{N}$, one gets that $\hat{C}_{-} \approx -1$ and $\hat{C}_{+} \approx 3144$. Therefore, our estimation for structural resistance parameter corresponds to $\hat{C}_{+}$ value, since $\hat{C}_{-}$ is negative.

Note that the value of both $\hat{t}$ and $\hat{L}$ used here and in the next section to estimate the structural resistance $C$ are chosen arbitrarily. The direct calculations with the exact values for physical parameters are intended to show that using the obtained formulas one can easily estimate the value of structural resistance. By fixing $\hat{L}$, the time $\hat{t}$ needed by water to reach $\hat{L}$ depends in general on different factors such as wood species or age of wood, and should be determined during the experiment.

### 2.1.6 Estimating structural resistance parameter (inertia neglected)

After disregarding the inertia term in (13), the governing equation can be rewritten in a non-dimensional form

$$\mathcal{H}\frac{d\mathcal{H}}{ds} + \mathcal{H} - \mathcal{H}_0 = 1. \tag{26}$$

The above equation can be solved by the method of separation of variables to get

$$-\mathcal{H} - (1 + \mathcal{H}_0)\ln\left(1 + \mathcal{H}_0 - H\right) = s. \tag{27}$$

The solution can't be written in the form of an explicit function $\mathcal{H}(t)$, but it is nevertheless possible to obtain useful results from that form. By substituting $s = t/T$ and $\mathcal{H} = H/L$ and also using the exact values for $T$ and $L$ the equation (27) becomes

$$-\frac{\hat{L}\rho g r}{2\gamma\cos(\theta)} - \left(1 + \frac{h_0\rho g r}{2\gamma\cos(\theta)}\right)\ln\left(1 + \frac{(-\hat{L} + h_0)\rho g r}{2\gamma\cos(\theta)}\right) = \frac{\hat{t}\rho^2 r^3 g^2}{16\mu\gamma(1+\hat{C})}. \tag{28}$$

Finally,

$$\hat{C} = \frac{1}{-\frac{\hat{L}\rho g r}{2\gamma\cos(\theta)} - \left(1 + \frac{h_0\rho g r}{2\gamma\cos(\theta)}\right)\ln\left(1 + \frac{(-\hat{L}+h_0)\rho g r}{2\gamma\cos(\theta)}\right)} \cdot \frac{\hat{t}\rho^2 r^3 g^2}{16\mu\gamma} - 1. \tag{29}$$

Taking the same end value condition as in the previous section, i.e. $\hat{t} = 1\,\text{day} = 86400\,\text{s}$ and $\hat{L} = 0.1\,\text{m}$ and using the same values for physical quantities as before one finally gets that $\hat{C} \approx 3011$.

Notice that the approximate value of the structural resistance that was obtained using formula (29) is approximately equal to the one calculated based on the reduced governing equation with the neglected gravitational term (see Section 2.1.5). However, it is worth to mention that for different values of $\hat{t}$ and $\hat{L}$ the difference between these two approximations may increase. Nevertheless, taking into account the ranges of time when the appropriate approximation is valid and because the $\hat{t}$ may be quite large in applications, the Authors expected that the formula (29) for estimating $\hat{C}$ may be more accurate.

### 2.1.7 Numerical simulation of model in wood

In this paper, the vessels are assumed to have a radius of $25\,\mu$m and structural resistance of 1000; the tracheids are assumed to have a radius of $10\,\mu$m and structural resistance of 1000. The radius of the tree trunk is estimated at 1 m.

The model of capillary rise in wood is implemented through Python. Using integration to solve the system of differential equations in the model, one can obtain the height and volume of water in the wood against time. Under standard temperature and pressure, the water in vessels reaches its Jurin's height of approximately 0.6 m in roughly a month; and about 15 months later, water in tracheids also reaches a Jurin's height about 1.4 m.

In the meantime, we estimate the water content in all tracheids or all vessels by assuming a density of them and summing up the water volume in each xylem. Here the density of vessels is set to be $5 \cdot 10^6\,\mathrm{m}^{-2}$, and that of tracheids to be $2 \cdot 10^9\,\mathrm{m}^{-2}$. The total water content reaches an equilibrium then as well.

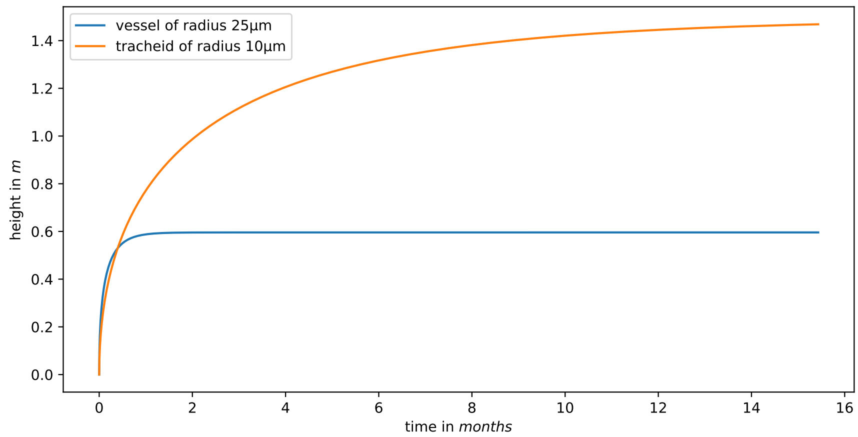


Figure 3: Numerical simulation of water height in wood

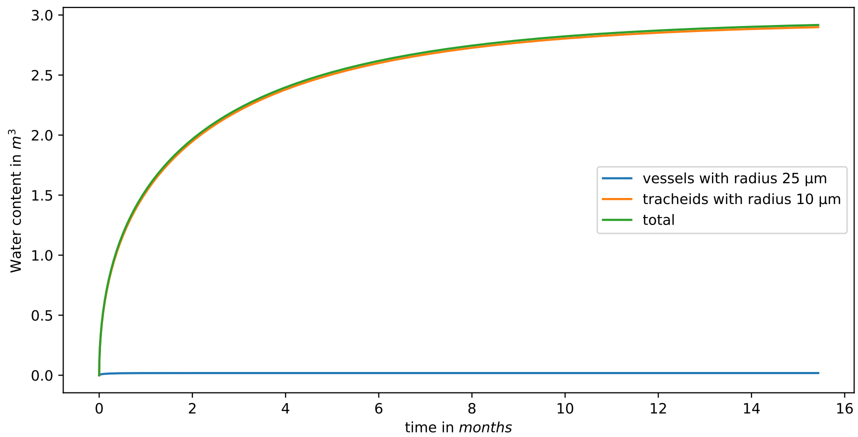


Figure 4: Numerical simulation of water volume in wood

## 2.2 Capillary moisture uptake in trees

### 2.2.1 Electric analogy

Roots absorb water, the stem of the tree transports water upwards through the xylems, and at the crown of the tree, the leaves constantly evaporate water, the amount depending on various variables e.g. temperature, light, etc. The model works on the assumption that the internal structure of the model tree is composed of narrow tubes and that the conduits are permeable from bottom to top.

Van den Honert (1948) viewed the flow of water in the plant as a catenary process in his quantification of the C-T theory. Given only the passive, without the active part (secretion), of the transpiration stream, *"each catena element is viewed as a hydraulic conductance (analogous to an electrical conductance) across which water (analogous to electric current) flows*(Tyree, 1997).

| | |
|---|---|
| tree height | $\Delta H$ |
| function of height over time | $h(t)$ |
| suction tension caused by an flux of evaporation generated at the tree height | $P_0 = \frac{AE\Delta H}{K}$ in [Pa] |
| transpiration (xylem pressure) | $\mathcal{T} = \frac{AE(h(t)-\Delta H)}{K} + P_0 = \frac{AE}{K}h(t)$ |
| wood conductivity | $K$ in $[\frac{\text{kg}\cdot\text{m}}{\text{s}\cdot\text{Pa}}]$ |
| area | $A$ in $[\text{m}^2]$ |
| flux of evaporation density | $E$ in $[\frac{\text{kg}}{\text{m}^2\cdot\text{s}}]$ |

According to Van den Honert (1948), as a result of the analogy to the electric circuit and Ohm's law, in particular, a prediction arises, that is, that continuous increase in $\mathcal{T}$ towards sap flow is the impelling cause of sap ascent (Tyree, 1997).

As per the prediction, the pressure gradient, expressed as $d\mathcal{T}/dh$, is proportional to the flux density of evaporation from leaves $E$ along the cross-section of the transpiration stream, i.e. $d\mathcal{T}/dh = AE/K$, where $A$ is a leaf surface from which water evaporates (Tyree, 1997). In the employed model, the flux of evaporation density $E$ is constant. However, if $E$ changes periodically over time, the results of the analysis will change as well. It is for practical reasons imperative to find an appropriate numerical analysis with $E = E(t)$. In subsequent sections, such an analysis can be found. To find an exact expression for $\mathcal{T}$ integrate equation $d\mathcal{T}/dh = AE/K$ both sides on the range $[h, \Delta H]$ with the auxiliary condition $\mathcal{T}(\Delta H) = P_0$. Indeed,

$$\int \frac{d\mathcal{T}}{dh}\, dh = \int \frac{AE}{K}\, dh \longrightarrow \mathcal{T}(h) = \frac{AE}{K}h + \text{const.},\ \mathcal{T}(\Delta H) = P_0, \tag{30}$$

where the constant of integration can be found by

$$\mathcal{T}(\Delta H) = \frac{AE}{K}\Delta H + \text{const.} = P_0 + \text{const.} = P_0, \tag{31}$$

what implies const. = 0. Finally, the expression for $\mathcal{T}$ is

$$\mathcal{T}(h) = \frac{AE}{K}h.$$

Taking $h(t) \leq \Delta H$ as a core assumption and transpiration that takes the form of $\mathcal{T} = AEh(t)/K$ *(see table above for details)*, the resulting equation is a derivation of the original formula, specifically the original formula in addition to the transpiration component (Tyree, 1997),

$$(\rho(h(t)+h_0)h'(t))' + (1+\hat{C})\frac{8\mu}{r^2}(h(t)+h_0)h'(t) + \rho g(h(t)) = \frac{2\gamma\cos(\theta)}{r} + \mathcal{T}(h). \tag{32}$$

Based on the analysis from the previous sections, it is reasonable to disregard the inertia term for sufficiently large periods of $t$ *(see section 2.3.4)* to get an accurate approximation of the exact solution. As a result, taking a closer look at the time needed to reach the top of the tree based on given specifications with $\zeta = AE/K$, the equation

$$(1+\hat{C})\frac{8\mu}{r^2}(h(t)+h_0)h'(t) + \rho g h(t) = \frac{2\gamma\cos(\theta)}{r} + \zeta h(t). \tag{33}$$

can be solved to get

$$\frac{g\rho\left(g\rho(h_0+L)\left(\log\left(\frac{h_0+L}{L}\right) - \log\left(\frac{h(\zeta-\rho g)}{\rho g L} + \frac{h_0}{L} + 1\right)\right) + h(\zeta - g\rho)\right)}{L(\zeta - g\rho)^2} = \frac{t}{T}, \tag{34}$$

where

$$L = \frac{2\gamma\cos\theta}{\rho g r}, \quad T = \frac{16\mu\gamma\cos(\theta)(1+\hat{C})}{r^3\rho^2 g^2}.$$

Assume that water rises from the bottom to the top through permeable capillaries. Suppose that $h = 20\,\text{m}$, $h_0 = 10^{-5}\,\text{m}$, $C = 1000$, $AE/K = 5\cdot 10^5\ \text{kg}\cdot\text{m}/\text{Pa}\cdot s$, and $r = 25\cdot 10^{-6}\,\text{m}$. The remaining parameters have already been used above in Section 2.3. After substituting given variables in eq. (34), results indicate that the water needs $t^* \approx 6$ days to reach a height of $h = 20\,\text{m}$. However, taking $r = 10^{-5}\,\text{m}$, the time $t^*$ significantly increases, $t^* \approx 37.5$ days. Fig. 5 reflects these results. Observe that the approximate time $t^*$ to reach the height of 20 m is quite long. $t^*$ depends mainly on the structural resistance $C$, tube radius $r$. Also, it depends strongly on the value of the $AE/K$ component. However, based on the performed analysis, the authors of this paper conjecture that in xylem conduits, in addition to capillary pressure and suction tension generated at the leaves surface, the flow may be accelerated by other processes accompanying the movement of water.

The transpiration is constant only when $h(t) = \Delta H$, so that $\mathcal{T} = P_0$, otherwise it is not constant since $h(t) - \Delta H \neq$ *const.* Replace the $\Delta H$ in $\mathcal{T}$ by the equilibrium height $h_e$ to find the Jurin's height. In other words, to determine Jurin's height, the assumption is that tension is generated directly at the equilibrium height. Hence, the Jurin's height is

$$h_e = \frac{1}{\rho g}\left(\frac{2\gamma\cos(\theta)}{r} + P_0\right). \tag{35}$$

The common values of $P_0$ are of the order of $10^6$ Pa, and for tall trees, $P_0 = 20$ atm $= 2.027 \cdot 10^6$ Pa (Canny, 1998), therefore, assuming that the rest of the parameters stay the same, Jurin's height is $h_e \approx 208.48$ m. Arguably, the pulling force generated by the leaves causes Jurin's height to be greater than the highest point of even the tallest trees.

An analogy to the electric circuit and Ohm's law, in particular, can be made where the current $I$ is analogous to the volumetric flow rate $Q$, the voltage potential difference $V$ corresponds to the pressure potential difference $\Delta p$ and the resistance $R$ is analogous to the drag at the tube $\mathcal{D}$. The table below shows the core analogies.

| Analogy | Hydraulic | Electric |
|---|---|---|
| resistance | drag $\mathcal{D}$ | resistance $R$ |
| quantity flux | volumetric flow rate $Q$ | current $I$ |
| potential | pressure $p$ in [Pa] $= [\frac{\text{kg}}{\text{m}\cdot\text{s}^2}]$ | voltage $V$ |

The Hagen-Poiseuille Law states that

$$\Delta\overline{p} = \frac{8\mu}{\pi r^4} LQ. \tag{36}$$

After transposing the equation to $Q$ and simplifying with circular cross-section area $A = \pi r^2$ and conductivity $K = r^2/8\mu$, the resulting equation can be further simplified with a mean drag $D = L/(K \cdot A)$, so that

$$\Delta Q = \frac{\pi r^4 \Delta p}{8\mu L} = K \cdot A \cdot \frac{\Delta p}{L} = \frac{\Delta p}{D}. \tag{37}$$

Therefore, a linear analogy becomes apparent in which $Q = \Delta p/\mathcal{D}$ is analogous to $I = V/R$.

By establishing the fact that the use of Hagen-Poiseuille's equation for cylindrical pipes is limited for xylem conduits and that the transpiration process in trees can be described by analogy to an electric circuit, the authors hypothesize that the structural resistance $C$ can vary for different wood species. Taking a closer look at real measures, for example, the radius of capillaries (vessels and tracheids), density, or the number of transport conduits in tree stem (refilling process), a difference in these parameters between wood species becomes apparent. An analogy to the electric circuit can therefore be pivotal to establish a point of comparison between different wood species to express a difference in real-world parameters. At this point, the authors believe that experimenting to determine a standard for further modeling is needed. To express the difference between wood species in a single variable, structural resistance $C$ in the employed model, by taking averages of experimentally determined values such as the time needed for water to fill "sufficiently similar" wood capillaries across species under controlled conditions- it is yet unknown whether such an experiment will yield meaningful results.

### 2.2.2 Numerical simulation of model in trees

Again, using Python, under the same group of parameters, the height and volume of water in trees are plotted against time. For a 20-meter high tree, after considering transpiration, water reaches the top of the tree in less than ten days, which is much closer to reality than with the wood model. After that, according to our model, the tree within its constant height stores a volume of water.

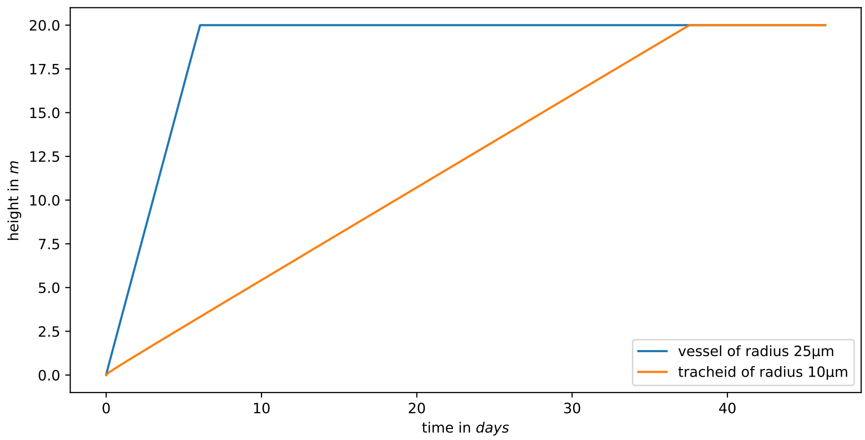


Figure 5: Numerical simulation of water height in a tree

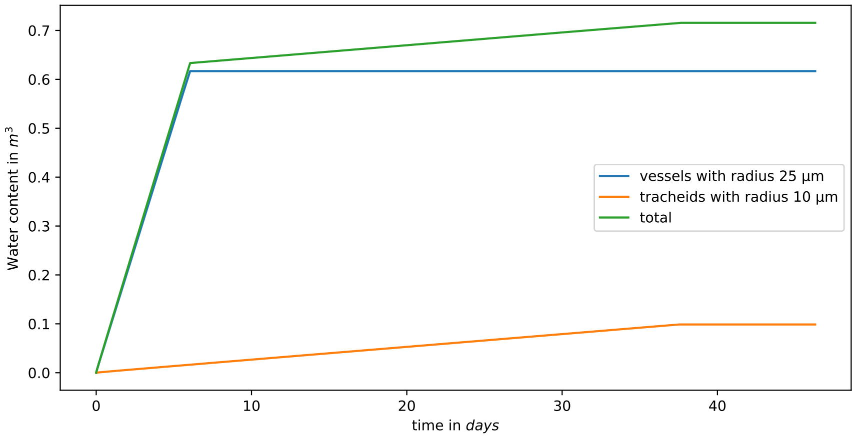


Figure 6: Numerical simulation of water content in a tree

#### 2.2.3 Model improvement based on periodic transpiration

The temperature appears to be periodic within a day, affecting the transpiration term $\mathcal{T}$. Since $A$ and $\Delta H$ reflect the trees' characteristics, they do not change according to temperature. We also assume $K$ to be a constant feature of pipe and water. Therefore, $E$ should be periodic. Taking that into account,

$$E = E(t),\ E(t) = E(t+T), \tag{38}$$

with $T$ being an appropriate period. Therefore, we let

$$E(t) = E_0 + \xi_1(t)\text{cos}(\alpha t) + \xi_2(t)\text{sin}(\alpha t), \tag{39}$$

where $\alpha = 2\pi/T$, $E_0$ is the mean value of $E$, $\xi_1(t), \xi_2(t)$ are two coefficient functions characterizing the temperature signal, and $T$ is 24 hours.

Finally, the governing system of equations modeling the capillary water uptake in trees with the flux of evaporation changing in time is as follows

$$\begin{cases} (\rho(h(t)+h_0)h'(t))' + (1+\hat{C})\frac{8\mu}{r^2}(h(t)+h_0)h'(t) + \rho g(h(t)) = \frac{2\gamma\cos(\theta)}{r} + \mathcal{T}(t), \\ \mathcal{T}'(t) = \frac{AE(t)}{K}h'(t). \end{cases} \tag{$T$}$$

The initial conditions for $h(t)$ are taken as before, i.e. $h(0) = 0$, $h'(0) = 0$, whereas initial value for the transpiration component is set to be equal zero, i.e. $\mathcal{T}(0) = 0$.

Under this periodic model setting, we observe the following simulated results for the same 20-meter tree.

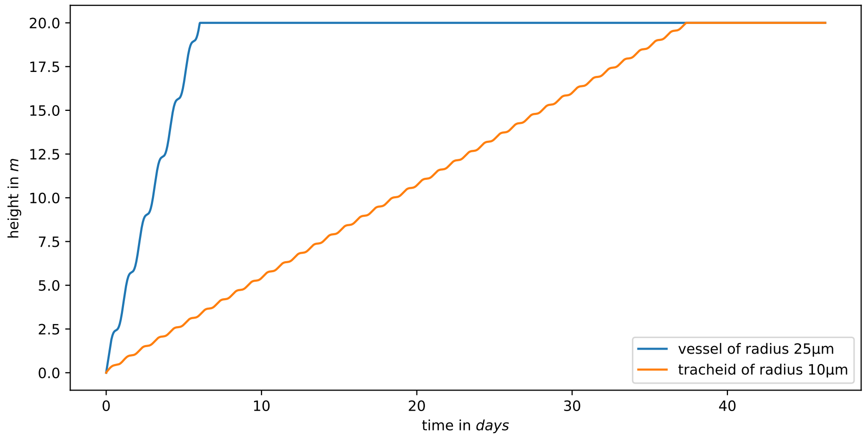


Figure 7: Numerical simulation of water height in a tree with periodic temperature

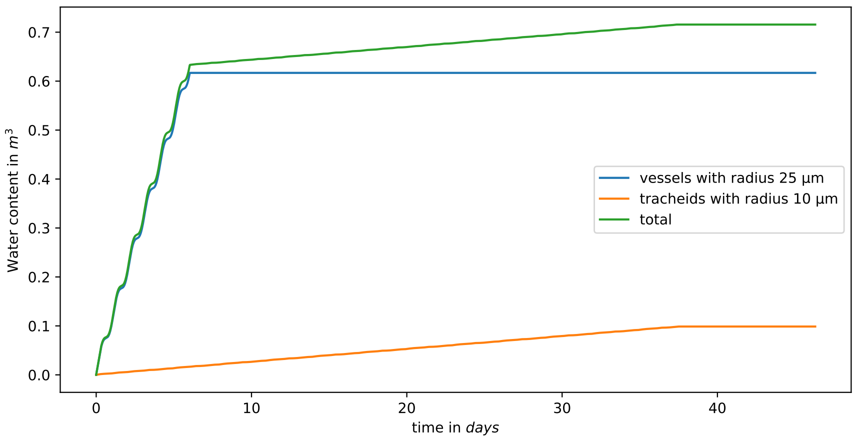


Figure 8: Numerical simulation of water content in a tree with periodic temperature

In the setting, water is always rising in the tree, while the periodic temperature has its daily impact on the rising speed. The eventual height and water content are not different from the simple version of the tree model, and the time it takes for the water to reach the top roughly remains the same as before.

## 2.3 Moisture balance in forest

In the forest system, the water enters through rainfall and exits through evaporation from leaves and soil. Then the net increase in water volume $[m^3]$ is the difference between precipitation and evaporation. In equilibrium, when total rainfall is equal to full evaporation, the forest stores a constant amount of water. However, when rainfall is less than evaporation, there is a net decrease in the total water amount in the forest. By mass conservation law, the water content in trees decreases by the same amount.

For the $i$-th tree, the volume of evaporated water $Tr[\mathrm{m}^3]$ during a period $[0, t]$ satisfies

$$\rho Tr_i(t) = \int_0^t A_i E(\tau)\mathrm{d}\tau, \tag{40}$$

where $\rho[\mathrm{kg/m}^3]$ is the density of water, $A_i[\mathrm{m}^2]$ is the total area of leaves in the tree $i$ and $E[\mathrm{kg}/(\mathrm{m}^2 \cdot \mathrm{s})]$ is the flux density of evaporation.

Therefore, the total water loss during $[0, t]$ from all trees is

$$\overline{Tr}(t) = \sum_{i=1}^{\infty} Tr_i(t). \tag{41}$$

In the same period $[0, t]$, expressing the volume of water evaporated from the floor of the forest as $S(t)$, and the volume of rainfall to the whole forest as $Rf(t)$, then the water deficit at every point of time $WD(\cdot)$ of the forest satisfies

$$\int_0^t WD(\tau)\mathrm{d}\tau = \overline{Tr}(t) + S(t) - Rf(t), \tag{42}$$

or

$$WD(t) = \overline{Tr}'(t) + S'(t) - Rf'(t). \tag{43}$$

When there has not been rainfall for a time interval $[0, t]$ so long that $\int_0^t WD(\tau)\mathrm{d}\tau = \overline{Tr} > 0$, i.e., the soil is dry enough and evaporation is negligible, the forest suffers from a drought. In this case, only evaporation from trees with a speed of $E$ causes total water loss in the forest. Assuming all trees are filled with water, then a change of water content over time in the trees can be plotted when there is no external water supply from the soil, and the maximum time for trees to survive can be solved (numerically) given a minimum threshold of water content inside the trees.

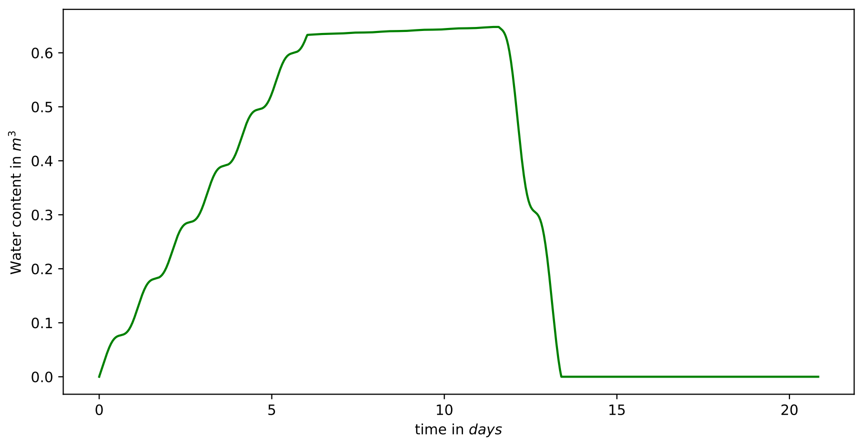


Figure 9: Numerical simulation of water content in a tree when drought occurs

Based on the previous periodic tree model of water content, and assuming that the drought occurs when the tree is thoroughly filled with water, i.e. the water supply in the soil for rising vanishes within a short time, we obtain the following plot. For the first 10 days, water in the tree just gradually rises to the top as before. After the drought in the forest, the water originally in the tree evaporates as time goes by. If there is no rainfall, the water content could decrease to 0 as shown above.

# 3 Discussion and conclusions

In Section 2.1, the subject matter is the model of capillary moisture uptake in wood. The model is proposed based on the capillary rise phenomenon, assuming that wood is composed of many narrow vertical tubes of different diameter. As it was underlined at the beginning of the report, the modelling of the water transport in plants differs from that in ordinary cylindrical pipes. Indeed, the internal structure o tree composed of the permeable wood tissues, the xylems, obliges to consider an additional resistance force caused by the structure of pits that connect consecutive xylem conduits. Thus to move from roots to leaves the water will have to pass through thousands of pits, where may be subjected to additional resistance, as opposed to one long cylindrical pipe where pits are not present. Hence, the structural resistance parameter that represents this structural property of wood, is introduced. However, the exact value of the structural resistance factor for the particular type of wood is not known and should be determined experimentally. To estimate the structural resistance parameter, surmise that for certain periods, two regimes are distinguishable. In the first regime, assume that it is possible to disregard gravitational forces for moisture uptake in a short pipe. In the second regime, consider reaching equilibrium height and ignore the inertial force. Therefore, the gravitational force is necessary to stop the uptake of water. The reduction of the main equation was justified by introducing appropriate dimensionless quantities and comparing the influence of the individual components in the model. By integrating the reduced equations and using experimental data, the estimates of the structural resistance factor can be easily found in appropriate regimes. The corresponding numerical analysis has been performed by simulating the change in water volume inside a wood sample that is vertically immersed in water. The structural properties of wood may offer useful insights into material science or sustainable architecture.

In section 2.2, based on the cohesion-tension theory, after considering the effect of transpiration to the wood model, the authors propose a model for capillary moisture uptake in trees. An analogy to the electric circuit, in particular to Ohm's law, is considered. The authors hypothesize that this analogy allows for an estimate of moisture content in different tree species by the structural resistance. For certainty, conducting more experiments and creating more complex modeling is needed.

Scaling up from trees to a forest ecosystem, to estimate the amount of moisture within a forest, knowing the base parameters of tree species within the ecosystem is imperative. This knowledge might provide insight into the health of a forest from a water economy point of view.

From a more macroscopic perspective, the employed model for moisture balance in the forest considers all trees in a forest, rainfall, and water storage in soil, basically the environmental conditions (ecosystem) of a given forest. This knowledge offers insight into the way that water volume in trees impacts drought and thus a closer look at the inner workings of ecosystems. The given model indeed offers a helpful approach to the estimation of hard to observe measures such as water volume inside the trees.

## 4 Acknowledgment

We would like to thank Danda Niranjan Kumar from the University of Koblenz-Landau and Mahmoud Elimam from Mansoura University for their valuable cooperation during the first Virtual ECMI Modelling Week. Some of the ideas presented in the report arose during the intense group discussions in which Danda Niranjan Kumar and Mahmoud Elimam actively participated.

## 5 Group work dynamics

Group work dynamics were generally satisfactory. Through a functioning operational structure that is the separation of the problem into three sub-problems that allowed the distribution of tasks within the team, work could progress swiftly and efficiently. One pair was mainly considering the model of capillary moisture uptake in wood. Another team member was working on capillary moisture uptake in trees using an analogy to electricity. The last member was considering the moisture balance in the forest. Meetings have been conducted regularly via MS Teams.

Every participant dully and amiably contributed. Tasks were usually completed in time so that there was no significant delay in successive pieces of work. All assignments reached completion through an avid exchange with the instructor.

People of vastly different backgrounds made up the team. That was only partly reflected within the various nationalities that are *Burkinabe, Chinese, Egyptian, German, Indian, Polish, Russian*, and *Serbian* and time zones present. The difference in time zones caused a challenge to organize regular meetings. Ultimately, almost every second Saturday became the meetup day for all team members. Although two team members could not take part in authoring this report due to corona-related matters, their contributions during the Modeling week stay appreciated.

The Corona-virus caused unprecedented challenges to the completion of the project, challenges that were only compounded by language difficulties, online-only communication, and different time zones. Nevertheless, it was still possible to communicate and work together to create a presentation after the Modeling week and write this final report.

# 6 Instructor's assessment

The problem of capillary moisture uptake in wood proves to be a complex and demanding issue as it requires extensive knowledge about the internal structure of wood as well as a deep understanding of the flow dynamics inside permeable xylem conduits. To fully evaluate the models, relevant experimental data is needed. However, due to the variety of side processes involved in water movement in wood tissues, that data is quite hard to collect. Despite these difficulties, the group worked diligently during the Virtual Modelling Week and after it, preparing the presented report. It is worth noting that the group's work included such tasks as literature review, mathematical models of the water uptake in wood and trees, simplified model of water balance in forests, numerical analysis of the proposed models. Because the participants had different educational backgrounds and diverse knowledge of fluid dynamics, some of the ideas were discussed in front of the group to work out the best solution to a given part of the problem. Group members efficiently divided tasks among themselves. Hence, they could obtain various compelling results.
The resulted models that describe the capillary moisture uptake in wood and trees are more than satisfactory and may be useful in a more detailed examination of the internal structure of wood. Furthermore, the model of water balance in forests may be thought-provoking from the ecological point of view and can be improved if relevant experimental data are available. Finally, all results and prepared report demonstrate the pronounced commitment of the participants during the Virtual Modelling Week, as well as their great potential in the field of mathematical modeling.

# References

P. Bentley and W. Blumer. Uptake of water by the lizard, moloch horridus. *Nature*, (194):699–700, 1962.

Martin J Canny. Transporting water in plants: Evaporation from the leaves pulls water to the top of a tree, but living cells make that possible by protecting the stretched water and repairing it when it breaks. *American Scientist*, 86(2):152–159, 1998.

Henry H Dixon and J Joly. On the ascent of sap. *Proceedings of the Royal Society of London*, (57):3–5, 1894.

Melvin Tyree and Martin Zimmermann. *Xylem Structure and The Ascent of Sap*. 01 2002. doi: 10.1007/978-3-662-04931-0.

Melvin T Tyree. The cohesion-tension theory of sap ascent: current controversies. *Journal of Experimental Botany*, 48(10):1753–1765, 1997.

TH Van den Honert. Water transport in plants as a catenary process. *Discussions of the Faraday Society*, 3:146–153, 1948.